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1983 J. Phys. A: Math. Gen. 16 3385

(http://iopscience.iop.org/0305-4470/16/14/030)

J. Phys. A: Math. Gen. 16 (1983) 3385–3392. Printed in Great Britain

Exact solution of the Poisson equation for a DC current on a saddle-shaped Helmholtz coil

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Received 12 January 1983, in final form 27 April 1983

Abstract. Explicit expressions for the vector potential associated with a DC current on a saddle-shaped Helmholtz coil have been evaluated. The Cartesian and cylindrical components of the magnetic field in a vacuum are derived from the vector potential. The solution for the components of the magnetic field can be written in closed form using elliptic integrals of the first and second kinds. The expressions may serve as a basis for the optimisation of various coil types used in high-resolution NMR spectroscopy and NMR tomography.

1. Introduction

Saddle-shaped coils have been in use in various fields of nuclear magnetic resonance (NMR) spectroscopy. The development of NMR imaging and NMR tomography has stimulated the search for new coil types which provide constant, pulsed, or alternating magnetic fields with certain homogeneity or gradient properties.

The magnetic field of a saddle-shaped coil has previously been calculated for points in and near the magnetic centre of the coil (Hoult 1978, Hoult and Richards 1976). Hoult (1978) investigated a description of the magnetic field in terms of an infinite series of spherical harmonics. Since the lowest-order terms for the variables r and z are of second order, higher-order terms have to be considered even if points quite near the centre of the coil will be calculated.

In the present investigation a closed expression for the magnetic field of a saddle-shaped Helmholtz coil in a vacuum for the case of direct currents is derived. The expressions derived are analytically exact. They do not involve expansions about the centre of the current configuration. The more general approach—derivation of $\boldsymbol{B}$ from explicit expressions of the vector potential $\boldsymbol{A}$—was used, so that alternating currents may also be considered.

The expressions for A_x and A_y become extremely clumsy if they are written explicitly in Cartesian coordinates. In many experimental situations, however, a representation of the components of $\boldsymbol{B}$ as a function of x, y and z is desired. Therefore, the components of $\boldsymbol{A}$ and $\boldsymbol{B}$ are expressed in Cartesian as well as cylindrical coordinates, the cylindrical coordinates being the more elegant for this problem.

2. Calculation of the vector potential

The magnetic field in a vacuum resulting from a current in a cylindrical saddle-shaped coil (figure 1) is evaluated with reference to the geometrical parameters of the coil: R_0 and Z_0 are the radius and half-height of the cylinder defining the geometry of the coil. The conductor is assumed to be infinitely thin and consists of two closed loops. The direction of the currents in the loops is such that the B field at $\boldsymbol{r}=0$ is parallel to the $+x$ axis (see figure 1). According to the superposition principle the magnetic field is calculated from Poisson's equation

$$\boldsymbol{A} = -4\pi \boldsymbol{j} \tag{1}$$

by adding the contributions of the four arcs at $z' = \pm Z_0$ and their vertical connections at $\varphi' = \pm\phi_0$ and $\pm\phi_0 + \pi$.

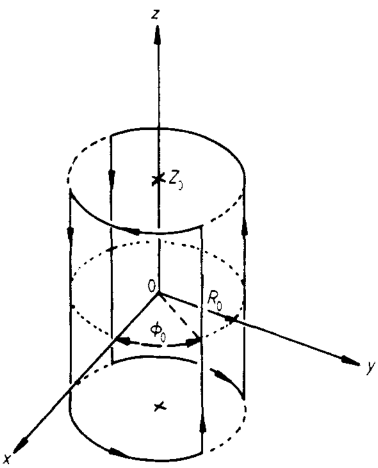

Figure 1. Geometry of the saddle-shaped Helmholtz coil with definition of geometrical parameters R_0, Z_0, ϕ_0 and direction of the currents.

The current of the saddle-shaped coil is represented by the current density $\Sigma\,(\boldsymbol{j}^{(a)} + \boldsymbol{j}^{(v)})$, $a, v = 1, 2, 3, 4$. The current density in the coil arcs is given by

$$j_x^{(a)} = -I \sin\varphi'\, \delta(r' - R_0)\, \delta(z' - Z_0^a), \tag{2a}$$

$$j_y^{(a)} = I \cos\varphi'\, \delta(r' - R_0)\, \delta(z' - Z_0^a), \tag{2b}$$

$$j_z^{(a)} = 0, \tag{2c}$$

with $-\phi_0 \leqslant \varphi' \leqslant \phi_0$, $Z_0^a = \pm Z_0$, and $\pi - \phi_0 \leqslant \varphi' \leqslant \pi + \phi_0$.

The current density in the vertical connections has the components

$$j_x^{(v)} = j_y^{(v)} = 0, \tag{2d}$$

$$j_z^{(v)} = I\delta(r' - R_0)\delta(R_0\varphi' - R_0\phi_0^v) \tag{2e}$$

with $\phi_0^v = \pm\phi_0$, $\pi \pm \phi_0$ and $-Z_0 \leqslant z' \leqslant Z_0$. The vector potential of the magnetic field is given by

$$\boldsymbol{A} = \sum_{a,s} \int \frac{\boldsymbol{j}^a(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{3}$$

The direction of the currents will be described by the limits of integration. The contribution of the first arc ($x'>0, z'>0$) to the x-component of the vector potential is derived using equations ($2a$), (3) and (4) for the integrations with respect to r' and z':

$$|\boldsymbol{r}-\boldsymbol{r}'|=[r^2-2rr'\cos(\varphi'-\varphi)+r'^2+(z-z')^2]^{1/2}. \tag{4}$$

The vector potential is therefore given by

$$A_x=-IR_0\int_{\phi_0}^{-\phi_0}\frac{\sin\varphi'\,\mathrm{d}\varphi'}{[r^2-2rR_0\cos(\varphi'-\varphi)+R_0^2+(z-Z_0)^2]^{1/2}}, \tag{5a}$$

$$A_y=IR_0\int_{\phi_0}^{-\phi_0}\frac{\cos\varphi'\,\mathrm{d}\varphi'}{[r^2-2rR_0\cos(\varphi'-\varphi)+R_0^2+(z-Z_0)^2]^{1/2}}. \tag{5b}$$

Substitution of φ' by $\pi-2x+\varphi$ yields

$$A_x=dk_2\int_{x_1}^{x_2}\frac{2\cos\varphi\sin x\cos x-\sin\varphi(\cos^2 x-\sin^2 x)}{(1-k_2^2\sin^2 x)^{1/2}}\,\mathrm{d}x, \tag{6}$$

with

$$d=IR_0^{1/2}r^{-1/2} \tag{7}$$

and

$$k_{1,2}^2=\frac{4rR_0}{(r+R_0)^2+(z\pm Z_0)^2}. \tag{8}$$

The integration has to be carried out in the limits of $x_1^{(1)}=\frac{1}{2}(\pi+\varphi-\phi_0)$ and $x_2^{(1)}=\frac{1}{2}(\pi+\varphi+\phi_0)$. Equation (6) can be expressed in terms of the elliptic integrals of the first and second kinds (Gradshteyn and Ryzhik 1965, Milne-Thomson 1965):

$$F(\phi,k)=\int_0^{\phi}(1-k^2\sin^2 x)^{-1/2}\,\mathrm{d}x, \tag{9a}$$

$$E(\phi,k)=\int_0^{\phi}(1-k^2\sin^2 x)^{1/2}\,\mathrm{d}x. \tag{9b}$$

The solution of equation (6) is then given by

$$A_x=-dk_2^{-1}\{2\cos\varphi(D_{22}-D_{21})+\sin\varphi[2E(x_2,k_2)$$
$$-2E(x_1,k_2)+(k_2^2-2)(F(x_2,k_2)-F(x_1,k_2))]\} \tag{10a}$$

with

$$D_{k,m}=(1-k_k^2\sin^2 x_m)^{1/2}. \tag{10b}$$

The y component of $\boldsymbol{A}$ associated with the current in the first arc ($x'>0, z'>0$) is evaluated using equation ($5b$):

$$A_y=dk_2^{-1}\{\cos\varphi[2E(x_2,k_2)-2E(x_1,k_2)+(k_2^2-2)(F(x_2,k_2)-F(x_1,k_2))]$$
$$-2\sin\varphi(D_{22}-D_{21})\}. \tag{11}$$

From equation ($2c$) follows:

$$A_z^a(r,\varphi,z)=0. \tag{12}$$

The components of the vector potential with respect to cylindrical coordinates are evaluated using the projection of $A(r, \varphi, z)$ on the coordinate $\boldsymbol{e}_r$:

$$A_r(r, \varphi, z) = A_x(r, \varphi, z) \cos \varphi + A_y(r, \varphi, z) \sin \varphi, \tag{13a}$$

and the component in the x–y plane perpendicular to $\boldsymbol{e}_r$ is given by

$$A_\varphi(r, \varphi, z) = -A_x(r, \varphi, z) \sin \varphi + A_y(r, \varphi, z) \cos \varphi. \tag{13b}$$

The vector potential of the arc can therefore be written as

$$A_r = -2dk_2^{-1}[(1-k_2^2 \sin^2 x_2)^{1/2} - (1-k_2^2 \sin^2 x_1)^{1/2}], \tag{14a}$$

$$A_\varphi = 2dk_2^{-1}\{E(x_2, k_2) - E(x_1, k_2) + (k_2^2/2 - 1)(F(x_2, k_2) - F(x_1, k_2))\}, \tag{14b}$$

$$A_z(r, \varphi, z) = 0. \tag{14c}$$

An independent calculation using $j_\varphi = I\delta(r' - R_0)\delta(z' - Z_0)$, $j_r = j_z = 0$, and

$$A_\varphi = \int \frac{j_\varphi(\boldsymbol{r}') \cos \alpha'}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}\boldsymbol{r}'$$

and

$$A_r = \int \frac{j_\varphi(\boldsymbol{r}') \cos(\alpha' + \pi/2)}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}\boldsymbol{r}'$$

led to identical results.

The vector potential resulting from the currents in the vertical parts of the conductor is determined by

$$\boldsymbol{j} = (0, 0, j_z) \tag{15a}$$

with

$$j_z = I\delta(r' - R_0)\delta(R_0\varphi' - R_0\phi_0). \tag{15b}$$

From equations (15) it follows that $A_x^v = A_y^v = A_\varphi^v = A_r^v = 0$, $v = 1, 2, 3, 4$. For the conductor at $\varphi' = \phi_0$ and with the direction of I as defined in figure 1 the vector potential is given by

$$A_z = -I[\ln(z - Z_0 + R_-) - \ln(z + Z_0 + R_+)]. \tag{16a}$$

R_{+-} is given by

$$R_{+-} = (r^2 - 2rR_0 \cos(\varphi - \phi_0) + R_0^2 + (z + - Z_0)^2)^{1/2}. \tag{16b}$$

In the following, the total vector potential due to the current I in the two loops is calculated. Summation of the terms $A^a + A^v$ yields in Cartesian coordinates:

$$A_x = -2d \sum_{i,j,k=1}^{2} (-1)^{i+j+k} k_k^{-1} \times [\cos \varphi D_{ijk} + \sin \varphi (E(x_{ij}, k_k) + (k_k^2/2 - 1)F(x_{ij}, k_k))], \tag{17a}$$

$$A_y = -2d \sum_{i,j,k=1}^{2} (-1)^{i+j+k} k_k^{-1} \times [\sin \varphi D_{ijk} - \cos \varphi (E(x_{ij}, k_k) + (k_k^2/2 - 1)F(x_{ij}, k_k))], \tag{17b}$$

$$A_z = -I \sum_{i,j,k=1}^{2} (-1)^{i+j} \ln[z - Z_k + ((x - X_j)^2 + (y - Y_i)^2 + (z - Z_k)^2)^{1/2}]. \tag{17c}$$

The following abbreviations have been used:

$$x_{ij} = (\phi_j + \varphi + (i-1)\pi)/2, \qquad X_j = (-1)^j X_0, \qquad Y_i = (-1)^i Y_0, \tag{18a, b, c}$$

$$Z_k = (-1)^k Z_0, \qquad \phi_j = (-1)^j \phi_0, \qquad D_{ijk} = (1 - k_k^2 \sin^2 x_{ij})^{1/2}. \tag{18d, e, f}$$

In order to write A_x, A_y and A_z consistently in Cartesian coordinates, the following substitutions have to be made in (17)

$$R_0 = (X_0^2 + Y_0^2)^{1/2}, \qquad r = (x^2 + y^2)^{1/2}, \tag{19a, b}$$

$$\phi_0 = \sin^{-1}[Y_0/(X_0^2 + Y_0^2)^{1/2}], \qquad \sin\varphi = y/(x^2+y^2)^{1/2}, \tag{19c, d}$$

$$\cos\varphi = x/(x^2+y^2)^{1/2}, \qquad k_k^2 = 4rR_0[(r+R_0)^2 + (z - Z_k)^2]^{-1}. \tag{19e, f}$$

The expressions for the components of $\boldsymbol{A}$ in cylindrical coordinates can be written in a much more compact form:

$$A_r = -2d \sum_{i,j,k=1}^{2} (-1)^{i+j+k} k_k^{-1} D_{ijk}, \tag{20a}$$

$$A_\varphi = 2d \sum_{i,j,k=1}^{2} (-1)^{i+j+k} k_k^{-1} [E(x_{ij}, k_k) + (k_k^2/2 - 1) F(x_{ij}, k_k)], \tag{20b}$$

$$A_z = -I \sum_{i,j,k=1}^{2} (-1)^{i+j+k} \ln[z - Z_k + 2(rR_0)^{1/2}(k_k^{-2} - \cos^2 x_{ij})^{1/2}]. \tag{20c}$$

3. Magnetic field

The components of $\boldsymbol{B}$ in cylindrical coordinates were derived using standard formulae for $\boldsymbol{B} = \operatorname{rot} \boldsymbol{A}$ in cylindrical coordinates:

$$B_r = r^{-1}(\partial A_z/\partial\varphi) - (\partial A_\varphi/\partial z), \tag{21a}$$

$$B_\varphi = (\partial A_r/\partial z) - (\partial A_z/\partial r), \tag{21b}$$

$$B_z = r^{-1}[(\partial(rA_\varphi)/\partial r) - (\partial A_r/\partial\varphi)]. \tag{21c}$$

The derivation of the elliptic integrals was carried out using (1)

$$\partial E(\phi, k)/\partial k = k^{-1}(E(\phi, k) - F(\phi, k)),$$

$$\partial F(\phi, k)/\partial k = (1-k^2)^{-1} k^{-1} [E - (1-k^2)F - k^2 \sin\phi \cos\phi (1 - k^2 \sin^2\phi)^{-1/2}].$$

The result is

$$B_r = Ig \sum_{i,j,k=1}^{2} (-1)^{i+j+k} (z - Z_k) k_k [4rR_0 S_{ij} C_{ij} G_{ij} D'_{ijk} + 2F(x_{ij}, k_k)$$
$$+ (k_k^2 - 2)(k_k^2 - 1)^{-1}(k_k^2 S_{ij} C_{ij} D_{ijk}^{-1} - E(x_{ij}, k_k))], \tag{22a}$$

$$B_\varphi = Ig \sum_{i,j,k=1}^{2} (-1)^{i+j+k} (z - Z_k) k_k [-2D_{ijk}^{-1} - 2r(r + R_0(1 - 2C_{ij}^2)) G_{ij} D'_{ijk}], \tag{22b}$$

$$B_z = Ig \sum_{i,j,k=1}^{2} (-1)^{i+j+k} k_k \{-2rF(x_{ij}, k_k) + [(k^2(r + R_0) - 2r)/(k^2 - 1)]$$
$$\times (E(x_{ij}, k_k) - k_k^2 S_{ij} C_{ij} D_{ijk}^{-1})\}. \tag{22c}$$

The following abbreviations have been used:

$$g = (4r^{3/2}R_0^{1/2})^{-1}, \qquad S_{ij} = \sin x_{ij}, \qquad C_{ij} = \cos x_{ij}, \tag{22d, e, f}$$

$$G_{ij} = [(r+R_0)^2 - 4rR_0 \cos^2 x_{ij}]^{-1}, \tag{22g}$$

$$D'_{ijk} = (1 - k_k^2 \cos^2 x_{ij})^{-1/2}, \qquad D_{ijk} \text{ see } (18f). \tag{22h}$$

The Cartesian components of $\boldsymbol{B}$ were derived using (17). The derivations $\partial A_i/\partial x_m$ cannot be expressed in closed form, since $\partial E/\partial x$, $\partial E/\partial y$, $\partial F/\partial x$ and $\partial F/\partial y$ lead to an infinite series. These terms, however, cancel in the differences $\partial A_y/\partial x - \partial A_x/\partial y$, so that the Cartesian components of $\boldsymbol{B}$ can be represented by

$$\begin{aligned} B_x = I \sum_{i,j,k=1}^{2} (-1)^{i+k} & \left(\frac{(z-Z_k)}{[(x-X_j)^2+(y-Y_i)^2+(z-Z_k)^2]^{1/2}}\right) \\ & \times\left(\frac{(y-Y_i)}{(x-X_j)^2+(y-Y_i)^2} - \frac{y}{r^2} + \frac{k_k^2(k_k^2-2)}{4r^3R_0(k_k^2-1)}(xyX_j - x^2Y_i)\right) \\ & - I \sum_{i,j,k=1}^{2} (-1)^{i+j+k} g \frac{xk_k(z-Z_k)}{r}\left(\frac{k_k^2-2}{k_k^2-1}E - 2F\right), \end{aligned} \tag{23a}$$

$$\begin{aligned} B_y = -I \sum_{i,j,k=1}^{2} (-1)^{i+k} & \left(\frac{(z-Z_k)}{[(x-X_j)^2+(y-Y_i)^2+(z-Z_k)^2]^{1/2}}\right) \\ & \times\left(\frac{(x-X_j)}{(x-X_j)^2+(y-Y_i)^2} - \frac{x}{r^2} + \frac{k_k^2(k_k^2-2)}{4r^3R_0(k_k^2-1)}(xyY_i - y^2X_j)\right) \\ & - I \sum_{i,j,k=1}^{2} (-1)^{i+j+k} g \frac{yk_k(z-Z_k)}{r}\left(\frac{k_k^2-2}{k_k^2-1}E - 2F\right). \end{aligned} \tag{23b}$$

Substitution of r by $(x^2+y^2)^{1/2}$ in (22c) leads to B_z.

4. Polynomial expansion of $\boldsymbol{B}_r$

In the following, some limiting expressions, which avoid the elliptic integrals, are evaluated and compared to expressions derived previously (3). For small values of k and ϕ a trigonometric series of the elliptic integrals may be used

$$F(\phi, k) = 2\pi^{-1}\bar{K}\phi - \sin\phi\cos\phi[a_0 + (2a_1/3)\sin^2\phi + \ldots]. \tag{24}$$

$\bar{K}$ and a_m are functions of k and may be represented as a power series in k; for details see the appendix.

Inserting the trigonometric series and the power series for k^n in (22a) and summation yields

$$B_r = A_0[1 + R_0^2/B + r^2A_{2r} + z^2A_{2z} + r^2 f(\varphi)A_{2\varphi} + \ldots] \tag{25a}$$

with

$$A_0 = (8IZ_0 \cos\varphi \sin\phi_0)/R_0B^{1/2}, \tag{25b}$$

$$A_{2r} = -(1/R_0^2) - (1/2B) - (3R_0^2/2B^2) + (5R_0^4/B^3), \tag{25c}$$

$$A_{2z} = -(6R_0^2/B^2) + (15R_0^2Z_0^2/2B^3), \tag{25d}$$

$$A_{2\varphi} = (4/R_0^2) + (2/B) + (3R_0^2/2B^2) + (5R_0^4/2B^3), \tag{25e}$$

$$f(\varphi) = \cos^2\varphi \cos^2\phi_0 + \sin^2\varphi\,(\sin^2\phi_0 - 2\cos^2\phi_0), \tag{25f}$$

$$B = R_0^2 + Z_0^2. \tag{25g}$$

Inspection of (22*b*) gives that $B_\varphi = 0$ for $\varphi = 0$. The expansion of B_x at the origin is, therefore, given by (25*a*) with

$$A_0 = 8IZ_0 \sin\phi_0/R_0B^{1/2}, \qquad \text{and} \qquad f = \cos^2\phi_0. \tag{25h, i}$$

For $\phi_0 = \pi/3$ and $r = z = 0$, this result is identical to equation (10) of Hoult and Richards (1976) if the correct units are used in the expressions. The dimension of B is determined by the factor A_0. Using cgs units for the coil parameters and for the currents (10 amperes = 1 $\mathrm{cm}^{1/2}\,\mathrm{g}^{1/2}\,\mathrm{s}^{-1}$), the result is given in Gauss (1G = 1 $\mathrm{cm}^{-1/2}\,\mathrm{g}^{1/2}\,\mathrm{s}^{-1}$).

Acknowledgments

I am deeply indebted to Ms Rumana Desai for critical discussions and help in translation and to Professor Dr H Rüterjans for the support of this investigation.

Appendix. Series expansion of the incomplete elliptic integrals of the first and second kinds

For small values of k and ϕ, we may use the series (Gradshteyn and Ryzhik 1965)

$$F(\phi, k) \doteq 2\pi^{-1}\bar{K}\phi - \sin\phi\cos\phi\{a_0 + (2/3)a_1\sin^2\phi + [(2\times 4)/(3\times 5)]a_2\sin^4\phi + \ldots\}, \tag{A1}$$

$$E(\phi, k) = 2\pi^{-1}\bar{E}\phi + \sin\phi\cos\phi\{b_0 + (2/3)b_1\sin^2\phi + [(2\times 4)/(3\times 5)]b_2\sin^4\phi + \ldots\}. \tag{A2}$$

The complete elliptic integrals $\bar{E}$ and $\bar{K}$ as well as the coefficients

$$a_\nu = \sum_{n=\nu+1}^{\infty} \left(\frac{(2n-1)!!}{2^n n!}\right)^2 k^{2n}, \tag{A3}$$

$$b_\mu = \sum_{m=\mu+1}^{\infty} \left(\frac{(2m-1)!!}{2^m m!}\right)^2 k^{2m}/(2m-1) \tag{A4}$$

are functions of k. In the case of cylindrical coordinates k^{2n} may be expanded according to

$$k^{2n} = \left(\frac{4rR_0}{B}\right)^n \left(1 + \frac{2rR_0 - 2zZ_k}{B} + \frac{r^2+z^2}{B}\right)^{-n}, \tag{A5}$$

with $B = R_0^2 + Z_0^2$. For small r and z, the terms $y_1 = 2(rR_0 - zZ_k)/B$ and $y_2 = (r^2+z^2)/B$ are of first and second order, respectively. Therefore, the following series representation may be used

$$k^{2n} = \left(\frac{4rR_0}{B}\right)^n \left(1 + \sum_{\mu=1}^{\infty} c_\mu(n)(y_1+y_2)^\mu\right). \tag{A6}$$

For natural n the coefficients are given by

$$c_\mu(n)=(-1)^\mu[n(n+1)(n+2)\ldots(n+\mu-1)]/\mu! \tag{A7}$$

Therefore,

$$k^{2n}=\left(\frac{4rR_0}{B}\right)^n\left(1+\sum_{\mu=1}^{\infty}c_\mu(n)\sum_{\nu=0}^{\mu}\binom{\mu}{\nu}y_1^{\mu-\nu}y_2^{\nu}\right). \tag{A8}$$

The terms $y_1^{\mu-\nu}y_2^{\nu}$ are of the order $\mu+\nu$ in r and z. Replacing k^{2n} in (A3) and (A4) by (A8) and inserting (A3) and (A4) in (A1) and (A2) yields the elliptic integrals in the form of a power series in r and z.

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